1. *Let a, b, c be nonnegative real numbers, now two of which are zero. Prove that*
$$\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+a^2} \geq \frac{\sqrt{3(a^2+b^2+c^2)}}{2}.$$

Solution. Rewrite the inequality as
$$\sum_{cyc}\left(\frac{a^3+b^3}{a^2+b^2}-\frac{a+b}{2}\right) \geq \sqrt{3\sum_{cyc}a^2}-\sum_{cyc}a+\sum_{cyc}\frac{b^3-a^3}{a^2+b^2}$$

$$\Leftrightarrow \sum_{cyc}\frac{(a-b)^2(a+b)}{2(a^2+b^2)} \geq \sum_{cyc}\frac{(a-b)^2}{\sqrt{3\sum_{cyc}a^2}+\sum_{cyc}a}$$
$$+\frac{(a-b)(b-c)(c-a)\left(\sum_{cyc}a^2b^2+abc\sum_{cyc}a\right)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

Since $\sqrt{3\sum_{cyc}a^2} \geq \sum_{cyc}a$, it suffices to prove

$$\sum_{cyc}\frac{(a-b)^2(a+b)}{2(a^2+b^2)} \geq \sum_{cyc}\frac{(a-b)^2}{2\sum_{cyc}a}+\frac{(a-b)(b-c)(c-a)\left(\sum_{cyc}a^2b^2+abc\sum_{cyc}a\right)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

$$\Leftrightarrow \sum_{cyc}(a-b)^2\left(\frac{a+b}{a^2+b^2}-\frac{1}{a+b+c}\right) \geq \frac{2(a-b)(b-c)(c-a)\left(\sum_{cyc}a^2b^2+abc\sum_{cyc}a\right)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

$$\Leftrightarrow \sum_{cyc}(a-b)^2\cdot\frac{2ab+ac+bc}{a^2+b^2} \geq \frac{2(a-b)(b-c)(c-a)\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2b^2+abc\sum_{cyc}a\right)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

By the AM-GM Inequality, we have

$$\sum_{cyc}(a-b)^2\cdot\frac{2ab+ac+bc}{a^2+b^2}$$
$$\geq 3\sqrt[3]{\frac{(a-b)^2(b-c)^2(c-a)^2(2ab+ac+bc)(2bc+ab+ac)(2ac+bc+ba)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}$$

It remains to prove that

$$3\sqrt[3]{\frac{(a-b)^2(b-c)^2(c-a)^2(2ab+ac+bc)(2bc+ab+ac)(2ac+bc+ba)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}$$

$$\geq \frac{2(a-b)(b-c)(c-a)\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2 + abc\sum_{cyc} a\right)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

$$\Leftrightarrow 27\left[\prod_{cyc}(2ab+ac+bc)\right]\left[\prod_{cyc}(a^2+b^2)^2\right]$$

$$\geq 8\left[\prod_{cyc}(a-b)\right]\left(\sum_{cyc} a\right)^3\left(\sum_{cyc} a^2b^2 + abc\sum_{cyc} a\right)^3$$

Since

$$\prod_{cyc}(2ab+ac+bc) \geq 2\left(\sum_{cyc} ab\right)^3$$

and

$$\prod_{cyc}(a^2+b^2)^2 \geq \frac{64}{81}\left(\sum_{cyc} a^2\right)^2\left(\sum_{cyc} a^2b^2\right)^2$$

It suffices to show

$$\frac{16}{3}\left(\sum_{cyc} ab\right)^3\left(\sum_{cyc} a^2\right)^2\left(\sum_{cyc} a^2b^2\right)^2$$

$$\geq \left[\prod_{cyc}(a-b)\right]\left(\sum_{cyc} a\right)^3\left(\sum_{cyc} a^2b^2 + abc\sum_{cyc} a\right)^3$$

Now, note that

$$8\left(\sum_{cyc} a^2b^2\right)^2\left(\sum_{cyc} ab\right)^2 - 3\left(\sum_{cyc} a^2b^2 + abc\sum_{cyc} a\right)^3$$

$$= 8\left(\sum_{cyc} a^2b^2\right)^2\left(\sum_{cyc} a^2b^2 + 2abc\sum_{cyc} a\right) - 3\left(\sum_{cyc} a^2b^2 + abc\sum_{cyc} a\right)^3$$

$$= A\left(\sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right) \geq 0$$

where

$$A = 5\left(\sum_{cyc} a^2b^2\right)^2 + 12abc\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right) + 3a^2b^2c^2\left(\sum_{cyc} a\right)^2$$

It remains to prove that

$$2\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)^2 \geq \left[\prod_{cyc}(a-b)\right]\left(\sum_{cyc} a\right)^3$$

Without loss of generality, we may assume $a+b+c=1$. Setting $q = ab+bc+ca, r = abc$, then

$$(a-b)(b-c)(c-a) \leq \sqrt{(a-b)^2(b-c)^2(c-a)^2} = \sqrt{q^2-4q^3+2(9q-2)r-27r^2}$$

we have to prove

$$2q(1-2q)^2 \geq \sqrt{q^2-4q^3+2(9q-2)r-27r^2}$$

If $9q \leq 2$, then

$$2q(1-2q)^2 - \sqrt{q^2-4q^3+2(9q-2)r-27r^2} \geq q\left[2(1-2q)^2 - \sqrt{1-4q}\right] \geq 0$$

Since

$$2(1-2q)^2 - \sqrt{1-4q} = \left(\sqrt{1-4q} - \frac{1}{2}\right)^2 + \frac{1}{4}[2(1-4q)^2+1] \geq 0$$

If $9q \geq 2$, then

$$\begin{aligned}\sqrt{q^2-4q^3+2(9q-2)r-27r^2} &= \sqrt{\frac{4}{27}(1-3q)^3 - \frac{1}{27}(27r-9q+2)^2} \\ &\leq \sqrt{\frac{4}{27}(1-3q)^3}\end{aligned}$$

$$\begin{aligned}&\Rightarrow 2q(1-2q)^2 - \sqrt{q^2-4q^3+2(9q-2)r-27r^2} \\ &\quad\geq 2q(1-2q)^2 - \sqrt{\frac{4}{27}(1-3q)^3} = 2q(1-2q)^2 - \frac{2}{9}(1-3q)\sqrt{3(1-3q)} \\ &\quad\geq 2q(1-2q)^2 - \frac{2}{9}(1-3q) = \frac{8}{729}(9q-2)(81q^2-63q+13) + \frac{46}{729} > 0.\end{aligned}$$

The inequality is proved. Equality holds if and only if $a=b=c$.

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